

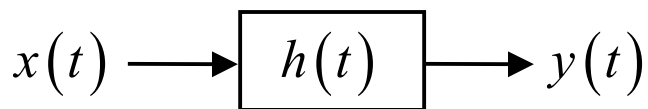
# III. Linear Transformations

- Properties of random sequence properties obtained as outputs to LTI systems
  - mean, cross-correlation between input/output sequences
- Frequency domain analysis (output PSD for LTI systems)
- Review of correlation matrix and eigendecomposition properties
- Optimal filtering (Part 1)
  - Orthogonality principle
  - FIR, IIR Wiener filter implementations
- Matched filter (Part 2)
  - Deterministic signal
  - Random signal

# III. Linear Transformations

- ◆ **Goal:** processing of stationary random sequences using LTI systems

Recall: Time domain analysis for LTI systems



$$y(t) =$$

$$y[n] =$$

- ◆ **Output random sequence properties**

- Mean:

$$E \{ y[n] \} =$$

- Input-output cross-correlation:

$$\begin{aligned} E \{ x[n] y^*[n - \ell] \} &= \\ &= \\ &= \end{aligned}$$



- Output correlation:

$$E \{ y[n] y^*[n - \ell] \} =$$

- Output covariance:

•**Example:** Given  $x[n]$  a RP with mean  $m_0$  and covariance  $C_x(l) = \sigma_0^2 \delta(l)$   
Compute the mean, correlation function, and covariance function of the output  $y[n]$  to the LTI system with impulse response  $h[n] = a^n u[n]$ ,  $|a| < 1$ .



- Frequency domain analysis: (output PSD for LTI systems)

Note:  $H(z) = \sum_n h(n)z^{-n} \rightarrow H^*\left(\frac{1}{z^*}\right) = \sum_n h^*(-n)z^{-n}$

- $S_{xy}(z) =$

- $S_{yx}(z) =$

- $S_y(z) =$

- For stable systems  $z = e^{j\omega}$  is within ROC

$$\Rightarrow S_{xy}(e^{j\omega}) =$$

$$S_{yx}(e^{j\omega}) =$$

$$S_y(e^{j\omega}) =$$

## ❖ Recall: Properties of Correlation Matrices

Let  $R$  be defined as the correlation matrix  $R$  for the stationary process  $\underline{x}$

(1)  $\lambda(R) \geq 0$

(2)  $R^k$  has eigenvalues  $(\lambda(R))^k$

(3) if  $R$  is a correlation matrix, then it has the following eigendecomposition

$$R = Q \Sigma Q^H \quad \begin{array}{l} \Sigma : \text{diagonal eigenvalue matrix} \\ Q : \text{unitary eigenvector matrix} \end{array}$$

(4) The eigenvectors are  $\perp$  to each other



## ❖ Matched Filter

- used to *detect* the presence of signals in additive noise (radar, communications, etc...)
- Two cases are considered:
  - (1) Signal is deterministic
  - (2) Signal is random

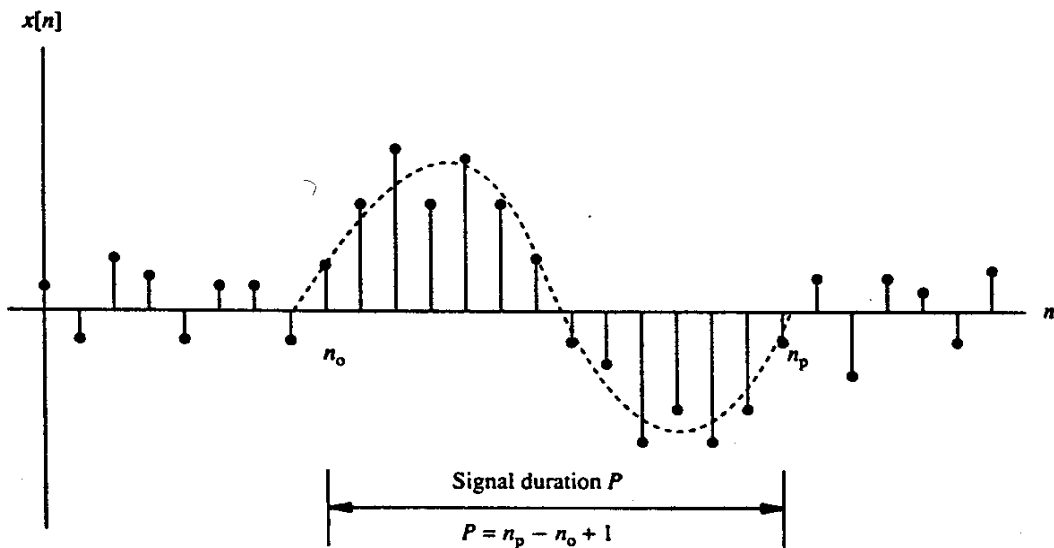


Figure 5.6 Finite-length signal observed in additive noise.

- $x[n]$  exists between  $[n_0, n_p]$
- noisy signal  $y[n]$  is defined as:
$$y[n] = x[n] + y_n[n]$$

❖ **Assume  $x[n]$  deterministic (additive noise)**

Goal: design the LTI filter

so that:

$$\text{SNR} = \frac{\left| y_s[n_p] \right|^2}{E \left\{ \left| y_n[n_p] \right|^2 \right\}} \quad \text{is maximized at } n = n_p$$

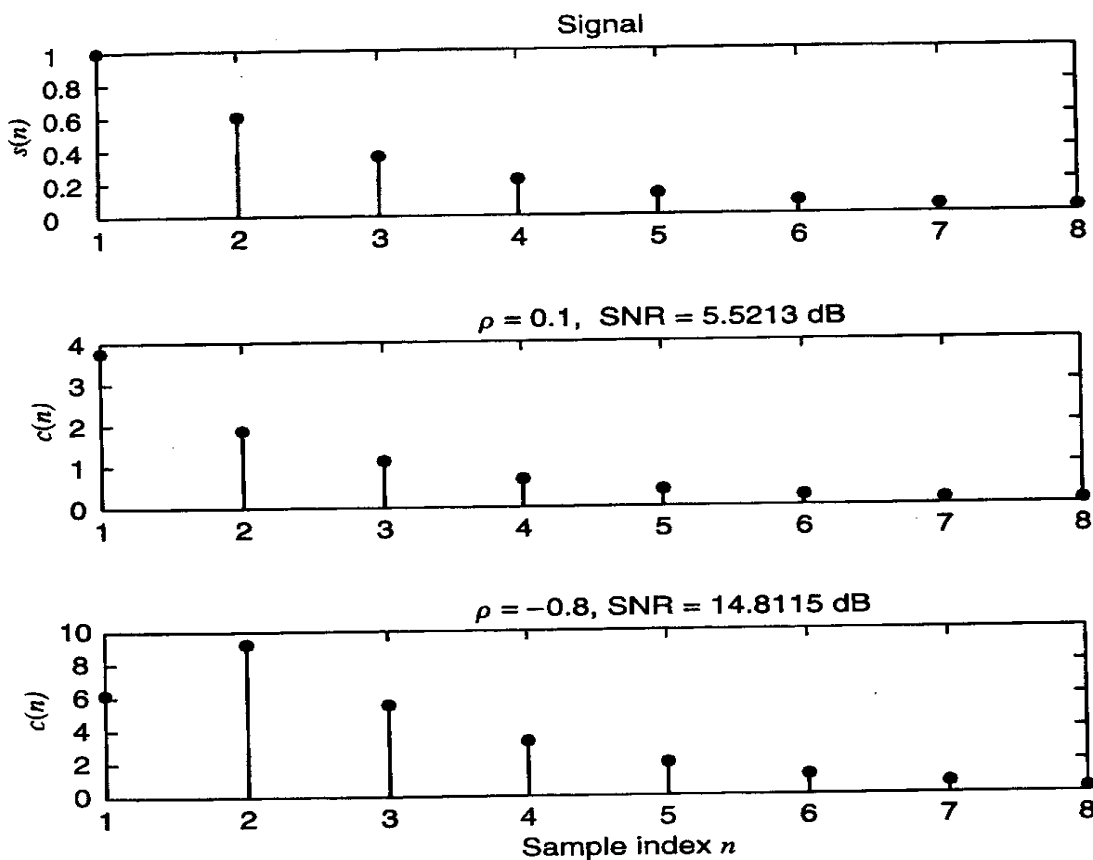
- $\underline{h} = [h[0], \dots, h[p-1]]^T$ ;  $\underline{x} = [x(n_0), \dots, x(n_p)]^T$

- $y[n] =$



**Example:**  $s[n] = a^n, \quad 0 \leq n \leq M-1 \quad |a| < 1$

Find matched filter coefficients and maximized SNR.



**FIGURE 6.35**

Signal and impulse responses of the optimum matched filter that maximizes the SNR in the presence of additive color noise.

**Comments:** Consider the finite-duration deterministic signal  $s(n)$  corrupted by additive noise  $v(n)$  with autocorrelation sequence  $r_v(k) = \sigma^2 \rho^{|k|} / (1 - \rho^2)$ . We determine and plot the impulse response of an  $M$ th-order matched filter for  $a = 0.6$ ,  $M = 8$ ,  $\sigma^2 = 0.25$ , and (a)  $\rho = 0.1$  and (b)  $\rho = -0.8$ .

Note that the signal vector is  $\mathbf{s} = [1 \ a \ a^2 \ \dots \ a^7]^T$  and that the noise correlation matrix  $\mathbf{R}_v$  is Toeplitz with first row  $[r_v(0) \ r_v(1) \ \dots \ r_v(7)]$ . The optimum matched filters are determined by  $\underline{\mathbf{h}} = \mathbf{R}_v^{-1} \underline{\mathbf{s}}_0$  and are shown above.

Note that for (1)  $\rho = 0.1$  the matched filter looks like the signal because the correlation between the samples of the interference is very small; that is, the additive noise is close to white,

(2)  $\rho = -0.8$  the correlation increases, and the shape of the optimum filter differs more from the shape of the signal. However, as a result of the increased noise correlation, the optimum SNR increases.

(2) Assume  $x(n)$  is random sequence (additive noise)

$$SNR = \frac{E \left\{ \left| y_s(n_p) \right|^2 \right\}}{E \left\{ \left| y_n(n_p) \right|^2 \right\}} =$$

- SNR is maximized when:

- **White noise case:**

- **Colored noise case**

**Numerical Note:**

You don't have to compute the entire eigendecomposition to get the maximum eigenvalue and its associated eigenvector

Proof: